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A duality involving Borel spaces

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ABSTRACT

The purpose of the paper is to show that there is a dual equivalence between sober Borel spaces and spatial Boolean σ-frames. We show that a discrete space is sober, if and only if, its cardinality is non-measurable and also show that many well known Borel spaces are sober.

1. Introduction

In this paper we show that there exists an adjunction between the category $\mathfrak{Bor}$ of Borel spaces and Borel maps and the opposite category $\mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}}$ of Boolean σ-frames and σ-frame homomorphisms (see Theorem 2.2). This in turn yields a dual equivalence (see Theorem 2.7) between the full subcategory $\mathfrak{BorSob}$ of sober Borel spaces (see page 212) and the full subcategory $\mathfrak{SpatBool}\sigma\mathfrak{Frm}$ of spatial Boolean σ-frames (see page 212). We provide characterisations of sobriety of Borel spaces (see Corollary 2.6) which for the special case of discrete Borel spaces yield a connection with non-measurable cardinals (see Theorem 3.6). This leads us to formulate a slogan: *under any usual universe of sets we can safely say that all discrete Borel spaces are sober*.

The adjunction discussed in this paper is very much similar to at least two well known ones:

(1) the adjunction between the category $\mathfrak{Top}$ of topological spaces and the opposite category $\mathfrak{Frm}^{\mathrm{op}}$ of frames as appears in the classical paper of Isbell [5] and is also well documented in the wonderful exposition of Johnstone [6],
(2) the adjunction between the category $\mathfrak{Alex}$ of Alexandroff spaces and the opposite category $\mathfrak{Reg}\sigma\mathfrak{Frm}^{\mathrm{op}}$ of regular σ-frames, as discussed in [2] or [3].

In fact the present adjunction can actually be seen as a restriction of the one in [2] or [3], provided the latter is suitably modified. To make the paper self-contained we recall a few relevant facts from [3]:

(1) An *Alexandroff* space is a pair $(X, \mathcal{Z})$ such that the following conditions are satisfied:
   (a) $\mathcal{Z}$ is closed under finite intersections and countable unions
   (b) $A, B \in \mathcal{Z}$ and $A \cap B = \emptyset \Rightarrow (\exists C, D \in \mathcal{Z})(A \cap C = \emptyset = B \cap D$ and $C \cup D = X$
   (c) for any $A \in \mathcal{Z}$ there exists a sequence $\{A_n\}_{n\geq 1}$ in $\mathcal{Z}$ such that $X \setminus A = \bigcup_{n\geq 1} A_n$
   (d) for each pair of distinct points x, y from X there exists a $A \in \mathcal{Z}$ such that A contains just one of x or y.

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If $(X, \mathcal{Z})$ and $(X', \mathcal{Z}')$ are Alexandroff spaces then a *coz-map* is a function $X \xrightarrow{f} Y$ such that $Z' \in \mathcal{Z}' \Rightarrow f^{\leftarrow}(Z') \in \mathcal{Z}$. Alexandroff spaces and coz-maps between them describe the category $\mathfrak{Alex}$.

(2) Given a σ-frame L and $a, b \in L$, it is said that a is *rather below* b and written $a \ll b$, if and only if there exists a $c \in L$ such that $a \wedge c = 0$ and $b \vee c = 1$. A σ-frame L is said to be *regular*, if and only if every element of L is a countable join of elements rather below it. Clearly, every σ-frame homomorphism preserve $\ll$, and thus we have the full subcategory $\mathfrak{Reg}\sigma\mathfrak{Frm}$ of regular σ-frames.

It is shown in [3] that there exists an adjunction $\mathfrak{Alex} \underset{\Psi}{\overset{\mathcal{U}}{\rightleftarrows}} \mathfrak{Reg}\sigma\mathfrak{Frm}$ with $\mathcal{U} \dashv \Psi$ and the description of the units, counits etc. are formally all exactly same as in the present paper. However, not every Borel space is an Alexandroff space as shown in Example 1.1; hence the adjunction as stated in [3] cannot be restricted to the present one.

Example 1.1. Let X be any set, $A \subseteq X$, $\mathcal{B}$ be the set of all the subsets P of X such that either $A \subseteq P$ or $P \cap A = \emptyset$. Clearly $\mathcal{B}$ is a Borel structure on X.

Thus, contrary to [3, Example (2), page 3], $\mathcal{B}$ is not an Alexandroff structure on X, showing that every Borel space need not be an Alexandroff space. However, every Borel space X in which the Borel structure separates points of X (see page 212) is indeed an Alexandroff space.

However, sometimes a larger category than $\mathfrak{Alex}$ is considered, namely the category $\mathfrak{Zero}$ whose objects are pairs $(X, \mathcal{Z})$ where $\mathcal{Z}$ satisfies all the the conditions like Alexandroff spaces except (1), and the maps are again similar to coz-maps. $\mathfrak{Zero}$ is indeed a good category — it is topological over $\mathfrak{Set}$(see [3, §1.7]) and $\mathfrak{Alex}$ is an epi-reflective subcategory of $\mathfrak{Zero}$. If the considerations of [3] however be raised to $\mathfrak{Zero}$ then the present adjunction can be seen to be a restriction of this new one.

To keep the pre-requisites to a minimum, we have included the definitions of most of the concepts used herein. However, the reader is assumed to have some familiarity with basic notions of categories, functors and adjunctions as can be found in [7]. The paper is organised as follows: the aim of §2 is to provide a description of the duality, Theorem 2.7; in the process we describe spatial Boolean σ-frames, sober Borel spaces and provide alternative characterisations for sobriety. In §3 we provide examples of spatial and non-spatial Boolean σ-frames, sober and non-sober Borel spaces, and finally show that a discrete Borel space is sober, if and only if its cardinality is non-measurable, Theorem 3.6. Finally in §4 we provide some directions for further investigations.

2. Description of the duality

A *Borel space* is a set X along with a set $\mathcal{B}X$ of subsets of X which is closed under countable unions and complements. The sets in $\mathcal{B}X$ are often referred to as the Borel subsets of X and $\mathcal{B}X$ itself is called the Borel structure on X; if X and Y are Borel spaces a map $X \xrightarrow{f} Y$ is said to be a *Borel map*, if and only if $U \in \mathcal{B}Y \Rightarrow f^{\leftarrow}(U) \in \mathcal{B}X$; Borel spaces and Borel maps amidst them make the category $\mathfrak{Bor}$ of Borel spaces.

The Borel structure $\mathcal{B}X$ of a Borel space X has a well known algebraic structure, namely that of Boolean σ-frames. More specifically, a σ-frame is a bounded lattice L in which every countable subset has a join and finite meets distribute over countable joins. A Boolean σ-frame is a σ-frame in which every element is complemented.

If L and M are σ-frames then a function $L \xrightarrow{f} M$ which preserves finite meets and countable joins is said to be a *σ-frame homomorphism*. Note that a σ-frame homomorphism preserve the empty meet, i.e., the top element 1 and the empty join, i.e., the bottom element 0, as well. σ-frames and σ-frame homomorphisms make the category $\sigma\mathfrak{Frm}$ of σ-frames.

Since any σ-frame homomorphism between Boolean σ-frames preserve complements, the Boolean σ-frames and σ-frame homomorphisms between them make a full subcategory $\mathfrak{Bool}\sigma\mathfrak{Frm}$ of $\sigma\mathfrak{Frm}$.

The structures are tailor made to the definition of the *Borel structure functor* $\mathfrak{Bor} \xrightarrow{\mathcal{B}} \mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}}$ which takes a Borel space to its Borel structure. Indeed there is another functor, namely the *Borel spectrum functor* in the reverse direction, which we now describe.

The lattice $\mathfrak{2}$ is a Boolean σ-frame and is the initial object of $\mathfrak{Bool}\sigma\mathfrak{Frm}$, so that the σ-frame homomorphisms from a Boolean σ-frame L to $\mathfrak{2}$ should be viewed as a point of L in $\mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}}$. The *spectrum* of a Boolean σ-frame L is the set ΓL of all the points of L in $\mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}}$ whose Borel structure is the set $\mathcal{B}\Gamma L = \{\Gamma_L(a) : a \in L\}$, where $\Gamma_L(a) = \{\theta \in \Gamma L : \theta(a) = 1\}$. If $L \xrightarrow{f} M$ be a σ-frame homomorphism between the Boolean σ-frames L and M then $\Gamma M \xrightarrow{\Gamma f} \Gamma L$ defined by $\Gamma f : \theta \mapsto \theta \circ f$ is a Borel map, since for any $a \in L$, $(\Gamma f)^{\leftarrow}(\Gamma_L(a)) = \Gamma_M(f(a))$. This describes the *Borel Spectrum functor* $\mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}} \xrightarrow{\Gamma} \mathfrak{Bor}$.

It would be worthwhile at this moment to consider different equivalent descriptions of a point of a Boolean σ-frame.

A *σ-prime filter* F on a lattice L is a proper filter (= an up-closed proper subset closed under finite meets) with the property that if for any sequence $\{a_n\}_{n \geq 1}$ of elements from L the filter F contains the join $\bigvee_{n \geq 1} a_n$ then the filter F contains at least one of the a_n's. In the special case of $\mathcal{B}X$, a filter $\mathcal{F}$ of Borel sets of a Borel space X is said to be *fixed*, if and only if $\bigcap \mathcal{F} \neq \emptyset$ and is said to be *free* otherwise.

Theorem 2.1. *For any σ-frame L there is a one-to-one correspondence between the points of L and the σ-prime filters of L. If further, L be Boolean, then the σ-prime filters on L are precisely the ultrafilters on L that are closed under countable meets. Finally, if $L = \mathcal{B}X$, then the $\{0, 1\}$-valued measures on X are precisely the points of $\mathcal{B}X$.*

Proof. If $L \xrightarrow{\theta} 2$ be a point of L then $\theta^{\leftarrow}(1)$ is a σ-prime filter of L and conversely, if S be a σ-prime filter of L then the map $L \xrightarrow{\sigma} 2$ defined by

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise} \end{cases}$$

is a point of L such that $\sigma^{\leftarrow}(1) = S$.

Now assume that L is Boolean and F be a σ-prime filter on L. Since for any $a \in L$, either $a \in F$ or $\neg a \in F$, but not both, it follows that F is an ultrafilter on L. Finally for any sequence $\{a_n\}_{n\geq 1}$ of elements of F, if $a = \bigwedge_{n\geq 1} a_n$ then since $1 = a \vee \neg a = a \vee \bigvee_{n\geq 1} \neg a_n \in F$, it follows that $a \in F$.

Conversely, let U be an ultrafilter on L which is closed under countable meets and $\{a_n\}_{n\geq 1}$ be a sequence of elements of L such that $a = \bigvee_{n\geq 1} a_n \in U$. Hence $\neg a = \bigwedge_{n\geq 1} \neg a_n \notin U$ which then implies that for some $n \geq 1$, $\neg a_n \notin U \Rightarrow a_n \in U$. Therefore U is a σ-prime filter on L.

Now let $L = \mathcal{B}X$ and $\mathcal{B}X \xrightarrow{\theta} 2$ be a point of $\mathcal{B}X$. For any countable family $\{A_n\}_{n\geq 1}$ of mutually disjoint Borel subsets of X, it easily follows $\theta(\bigcup_{n\geq 1} A_n) = \bigvee_{n\geq 1} \theta(A_n) = \sum_{n\geq 1} \theta(A_n)$, so that θ itself is a $\{0, 1\}$-valued measure on X.

Conversely, let $\mathcal{B}X \xrightarrow{\mu} \{0, 1\}$ be a two-valued measure on X and $\{A_n\}_{n\geq 1}$ be a countable family of Borel subsets of X. If:

$$B_n = \begin{cases} A_n \Big\backslash \left(\bigcup_{1\leq k<n} A_k\right), & \text{if } n \geq 2, \\ A_1, & \text{otherwise,} \end{cases}$$

then the family $\{B_n\}_{n\geq 1}$ is a mutually disjoint family of Borel subsets of X having the same union.

Hence $\mu(\bigcup_{n\geq 1} A_n) = \mu(\bigcup_{n\geq 1} B_n) = \sum_{n\geq 1} \mu(B_n)$, since μ is a two-valued measure, implying thereby that:

$$\mu(\bigcup_{n\geq 1} A_n) = \begin{cases} 0, & \text{if } n \geq 1 \Rightarrow \mu(B_n) = 0 \\ 1, & \text{otherwise} \end{cases}.$$

Now, $\mu(B_n) = 0$ for all $n \geq 1$ implies $\mu(A_1) = 0$ and for each $n > 1$, $\mu(A_n) = \mu(\bigcup_{1\leq k<n} B_k) = \sum_{1\leq k<n} \mu(B_k) = 0$; conversely $\mu(A_n) = 0$, for each $n \geq 1$ implies $\mu(B_n) = 0$ for all $n \geq 1$, since $B_n \subseteq A_n$. Therefore:

$$\mu(\bigcup_{n\geq 1} A_n) = \begin{cases} 0, & \text{if } n \geq 1 \Rightarrow \mu(A_n) = 0 \\ 1, & \text{otherwise} \end{cases}.$$

Hence, $\mu(\bigcup_{n\geq 1} A_n) = \bigvee_{n\geq 1} \mu(A_n)$.

Also, $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$ entails that $\mu(A_1 \cap A_2) = \mu(A_1) \wedge \mu(A_2)$.

Hence μ is a point of $\mathcal{B}X$. $\square$

Now, given a Borel space X, for each $x \in X$ the map $\mathcal{B}X \xrightarrow{\eta_X(x)} 2$ defined by $\eta_X(x)(U) = 1$, if and only if $x \in U$ is a point of $\mathcal{B}X$; furthermore, for any $U \in \mathcal{B}X$, $\eta_X{}^{\leftarrow}(\Gamma_U(\mathcal{B}X)) = U$, showing that the map $X \xrightarrow{\eta_X} \Gamma\mathcal{B}X$ is a Borel map. Also, given any Boolean σ-frame L and any Borel map $X \xrightarrow{f} \Gamma L$ the σ-frame homomorphism $L \xrightarrow{f^*} \mathcal{B}X$ defined by $f^*(a) = f^{\leftarrow}(\Gamma_a(L))$ is the unique σ-frame homomorphism which makes the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \Gamma\mathcal{B}X \arrow[d, "\Gamma f^*"] \\ & \Gamma L \end{tikzcd}$$

commute. Hence we get:

Theorem 2.2. $\mathfrak{Bor} \underset{\Gamma}{\overset{\mathcal{B}}{\rightleftarrows}} \mathfrak{Bool}\sigma\mathfrak{Frm}^{\mathrm{op}}$ *with $\mathcal{B}$ being left adjoint to Γ.*

Furthermore, the following facts are true for the adjunction:

(1) *the units are the Borel maps $X \xrightarrow{\eta_X} \Gamma\mathcal{B}X$ for each Borel space X, where for any $x \in X$, $U \in \mathcal{B}X$, $\eta_X(x)(U) = 1$, if and only if $x \in U$,*

(2) *the counits are the σ-frame homomorphisms $L \xrightarrow{\epsilon_L} \mathcal{B}\Gamma L$ for each Boolean σ-frame L, where $\epsilon_L(a) = \Gamma_L(a)$,*

(3) *for each Borel space X, the diagram*
$$\begin{tikzcd} \mathcal{B}X \arrow[r, "\epsilon_{\mathcal{B}X}"] \arrow[dr, equal] & \mathcal{B}\Gamma\mathcal{B}X \arrow[d, "\mathcal{B}\eta_X"] \\ & \mathcal{B}X \end{tikzcd}$$
commutes; in particular each $\epsilon_{\mathcal{B}X}$ *is a section (= injective* σ*-frame homomorphism) in* $\mathfrak{Bool}\sigma\mathfrak{Frm}$,

(4) *for each Boolean* σ*-frame L,*
$$\begin{tikzcd} \Gamma L \arrow[r, "\eta_{\Gamma L}"] \arrow[dr, equal] & \Gamma\mathcal{B}\Gamma L \arrow[d, "\Gamma\epsilon_L"] \\ & \Gamma L \end{tikzcd}$$
so that each $\eta_{\Gamma L}$ *is a section in* $\mathfrak{Bor}$.

The following is immediate for the counits:

Theorem 2.3. *Let L be a Boolean* σ*-frame and* $L \xrightarrow{\epsilon_L} \mathcal{B}\Gamma L$ *be a counit map.*
(1) ϵ_L *is surjective and hence an epimorphism of Boolean* σ*-frames.*
(2) ϵ_L *is injective, if and only if* ϵ_L *preserves the strict order of L.*

A Boolean σ-frame L is said to be *spatial*, if and only if ϵ_L is an isomorphism and $\mathfrak{SpatBool}\sigma\mathfrak{Frm}$ is the full subcategory of spatial Boolean σ-frames; each $\mathcal{B}X$ is a spatial Boolean σ-frame, Theorem 2.2 & Theorem 2.3, and the full subcategory $\mathfrak{SpatBool}\sigma\mathfrak{Frm}$ is an epi-reflective subcategory of $\mathfrak{Bool}\sigma\mathfrak{Frm}$ with ϵ_L being the reflection. However, not all Boolean σ-frames are spatial, as shown in Example 3.1.

The Borel structure $\mathcal{B}X$ of a Borel space X *separates points of* X, if and only if for all distinct pair of elements x, y from X there exists a pair of mutually disjoint Borel sets $U, V \in \mathcal{B}X$ such that $x \in U$, $y \in V$.[1] With this criterion, one easily sees that:

Theorem 2.4. *For a Borel space X,* η_X *is injective, if and only if* $\mathcal{B}X$ *separates points of X.*

A two-valued measure $\mathcal{B}X \xrightarrow{p} [0, 1]$ is said to be *concentrated at a point*, if and only if there exists a $x_0 \in X$ such that for all $U \in \mathcal{B}X$, $p(U) = 1 \Leftrightarrow x_0 \in U$.

Theorem 2.5. *The following are equivalent for a Borel space X:*
(1) *the unit* $X \xrightarrow{\eta_X} \Gamma\mathcal{B}X$ *is a surjection,*
(2) *every* σ*-prime filter on* $\mathcal{B}X$ *is fixed,*
(3) *the* σ*-prime filters on X are precisely of the form* $\mathcal{U}_x = \{U \in \mathcal{B}X : x \in U\}$, *for each* $x \in X$,
(4) *every two-valued measure* $\mathcal{B}X \xrightarrow{p} \{0, 1\}$ *is concentrated at some point of X.*

Proof
(1) ⇒ (2) Let $\mathcal{P}$ be a σ-prime filter on $\mathcal{B}X$. Using Theorem 2.1 let π be the corresponding point of $\mathcal{B}X$. From the hypothesis there exists a $x \in X$ such that $\eta_X(x) = \pi$; consequently, $x \in \bigcap\mathcal{P}$, implying $\mathcal{P}$ is fixed.
(2) ⇒ (3) Obviously each $\mathcal{U}_x$ is a fixed σ-prime filter on $\mathcal{B}X$. If $\mathcal{P}$ be a σ-prime filter on $\mathcal{B}X$ then from hypothesis, $\mathcal{P}$ is fixed. Hence there exists a $x \in X$ such that $P \in \mathcal{P} \Rightarrow x \in P$, i.e., $\mathcal{P} \subseteq \mathcal{U}_x$. But from Theorem 2.1, $\mathcal{P}$ is an ultrafilter, so that $\mathcal{P} = \mathcal{U}_x$.
(3) ⇒ (4) If $\mathcal{B}X \xrightarrow{p} \{0, 1\}$ be a two-valued measure on X then $p^{\leftarrow}(1)$ is a σ-prime filter on $\mathcal{B}X$. Hence $p^{\leftarrow}(1) = \mathcal{U}_x$, for some $x \in X$, implying p is concentrated at x.
(4) ⇒ (1) If $\theta \in \Gamma\mathcal{B}X$ be a point of $\mathcal{B}X$ then from Theorem 2.1, θ is a $\{0, 1\}$-valued measure on X. Hence θ is fixed at some element $x_0 \in X$, i.e., for any $U \in \mathcal{B}X$, $\theta(U) = 1 \Leftrightarrow x_0 \in U$. Therefore, $\theta = \eta_X(x_0)$, proving that η_X is a surjection. □

Corollary 2.6. *For any Borel space X,* $X \xrightarrow{\eta_X} \Gamma\mathcal{B}X$ *is an isomorphism in* $\mathfrak{Bor}$, *if and only if* $\mathcal{B}X$ *separates points of X and any one of the conditions of Theorem 2.5 hold good.*

Proof. Since for any $U \in \mathcal{B}X$, $U = \eta_X(\Gamma_U(\mathcal{B}X))$, it follows that η_X is an isomorphism, if and only if it is a bijection. □

We shall call a Borel space X to be *sober*, if and only if η_X is an isomorphism in $\mathfrak{Bor}$, and the full subcategory $\mathfrak{BorSob}$ of all sober Borel spaces is a reflective subcategory of $\mathfrak{Bor}$, with η_X being the reflection.

A consequence of the theory of adjunctions yield the duality:

[1] This is very much similar to the *normal separation axiom* for topological spaces, but indeed the presence of complements convert the T_1-like separation condition for Borel spaces into *normal*-like separation condition.

Theorem 2.7. *The category $\mathfrak{BorSob}$ of sober Borel spaces is dually equivalent to the category $\mathfrak{SpatBool}\sigma\mathfrak{Frm}$ of spatial Boolean σ-frames.*

3. Examples

It has already been established using Theorem 2.2 that for each Borel space X, $\mathcal{B}X$ is a spatial Boolean σ-frame. The following example, which appears in [3, page 53, (2)] shows that all Boolean σ-frames are not spatial:

Example 3.1. Let L be the collection of all regular open sets of $(0, 1)$. It is known that L is a complete Boolean algebra without any atoms. Further any σ-prime filter on L is completely prime.[2] Since completely prime filters on a complete Boolean algebra determine atoms, it follows that L has no non-trivial σ-prime filters. Hence using Theorem 2.1, ϵ_L cannot be injective. Consequently L is not a spatial Boolean σ-frame.

Theorem 3.2. *For any Boolean σ-frame L, the unit $\Gamma L \xrightarrow{\eta_{\Gamma L}} \Gamma\mathcal{B}\Gamma L$ is a surjection.*
In particular, the Borel spaces ΓL are sober.

Proof. From Theorem 2.2, the Borel maps $\eta_{\Gamma L}$ are always sections in the category $\mathfrak{Bor}$. In particular, they are injections.
It is enough to show that the Borel maps $\eta_{\Gamma L}$ are surjections and then the rest follows from Corollary 2.6.
Firstly observe for any $\theta \in \Gamma L$:

$$\eta_{\Gamma L}(\theta)\big(\Gamma_L(a)\big) = 1 \Leftrightarrow \theta(a) = 1$$
$$\Rightarrow \Gamma_L(a) \in \eta_{\Gamma L}(\theta)^{\leftarrow}(1) \text{ iff } \theta \in \Gamma_L(a).$$

Let $\mathcal{B}\Gamma L \xrightarrow{\xi} 2$ be a point of $\mathcal{B}\Gamma L$ and define the map $L \xrightarrow{\xi'} 2$ by

$$\xi'(a) = 1 \text{ iff } \Gamma_L(a) \in \xi^{\leftarrow}(1).$$

Since $\xi^{\leftarrow}(1)$ is a σ-prime filter on $\mathcal{B}\Gamma L$ it follows that ξ' is a point of L. Hence, $\xi = \eta_{\Gamma L}(\xi')$, completing the proof that $\eta_{\Gamma L}$ is a surjection. □

A Borel space X is said to be *separated*, if and only if $\mathcal{B}X$ is countably generated and contains all the singletons.

Theorem 3.3. *Every separated Borel space is sober.*

Proof. Let X be a separated Borel space, $\mathcal{B}X \xrightarrow{p} \{0, 1\}$ be a $\{0, 1\}$-valued measure on X and $\mathcal{A}$ be a countable generator of $\mathcal{B}X$. It is enough to show that p is concentrated at some point of X and then Corollary 2.6 concludes that X is sober.
Using Theorem 2.1, $\mathcal{P} = \{M \in \mathcal{B}X : p(M) = 1\}$ is an ultrafilter on $\mathcal{B}X$ closed under countable intersections. Let:

$$\mathcal{A}_0 = \{A \in \mathcal{A} : p(A) = 1\},$$
$$\mathcal{A}_1 = \{A \in \mathcal{A} : p(A) = 0\},$$

and

$$T = \Big(\bigcap \mathcal{A}_0\Big) \cap \Big(\bigcap\{A^c : A \in \mathcal{A}_1\}\Big).$$

Clearly, from the construction, $T \in \mathcal{P}$. Finally, since $\mathcal{B}X$ contains every singleton and is countably generated, it follows that T is a singleton, say $\{x\}$. Hence, p is concentrated at x. □

In particular, we have the following list of sober Borel spaces:

Corollary 3.4. *The following topological spaces with the Borel structure being generated by the topology are sober Borel spaces*:
(1) *any Euclidean space $\mathbb{R}^n$*;
(2) *any separable metric space*;
(3) *any T_1 topological space with a countable base for their topology.*

[2] A filter F in a complete Boolean algebra L is said to be *completely prime*, if and only if for all subsets $P \subseteq L$, $\bigvee P \in F \Rightarrow P \cap F \neq \emptyset$.

Every sober Borel space is not necessarily separated: consider any Boolean σ-frame L which is not countably generated and then ΓL is an example of a sober non-separated Borel space.

All Borel spaces, however are not sober:

Example 3.5. Let X be any uncountable set and $\mathcal{B}X$ be the set of all subsets of X that are either countable or their complements are countable.

It $\mathcal{B}X \xrightarrow{p} \{0,1\}$ be a $\{0,1\}$-valued measure on X then:

- If there exists a countable $A \subseteq X$ such that $p(A) = 1$, then there exists a $x \in A$ such that $p(\{x\}) = 1$.
- If every countable set $A \subseteq X$ has $p(A) = 0$, then the set $\mathcal{P} = \{A \in \mathcal{B}X : p(A) = 1\}$ is an ultrafilter closed under countable intersections and contains all co-countable sets, i.e., $\mathcal{P} \supseteq \mathcal{B} = \{X \setminus A : A \subseteq X$ and is countable$\}$. Hence, $\bigcap \mathcal{P} = \emptyset$, implying that the two-valued measure p is not concentrated at any point.

  Hence X is not a sober Borel space.

A set X is said to have a *measurable cardinal*, if and only if there exists $\{0,1\}$-valued measure on the discrete Borel space X which is not concentrated at any point of X. In other words, there exists a two-valued measure $\mathcal{P}(X) \xrightarrow{p} \{0,1\}$ such that $p(X) = 1$ and for every countable subset A, $p(A) = 0$. A set X whose cardinal is not measurable is said to have a *non-measurable* cardinal. Every countable cardinal is non-measurable, and whether every infinite cardinal is non-measurable is a celebrated unsolved problem. It follows from Corollary 2.6 that:

Theorem 3.6. *The discrete Borel space on a set X is sober, if and only if X has a non-measurable cardinal.*

4. Concluding remarks

Considering the introductory remarks, the duality in this paper is not completely new, although it might not have been explicitly present in the literature. However, there are some new features associated with the adjunction which we present here.

Firstly, consider the statement of Theorem 3.6. A similar statement holds for realcompactness[3]:

Theorem 4.1 (see [4,1]). *A discrete space is realcompact, if and only if its cardinal is non-measurable.*

Given the similarity of the present duality, we are not aware of any such similar result involving measurable cardinals so early in the context of frames[4] or σ-frames.

There is yet another place where there is a similarity between sober Borel spaces and realcompact topological spaces. In Theorem 2.1 we have established that the σ-prime filters on a Boolean σ-frame L are precisely those ultrafilters that are closed under countable intersections, and combining with Corollary 2.6 it follows that a Borel space X is sober, if and only if $\mathcal{B}X$ separates points and every ultrafilter on the Boolean σ-frame $\mathcal{B}X$ which are closed under countable intersections are fixed. Considering a topological space Z and the ring $C(Z)$ of all its real valued continuous functions the following facts are known:

(1) Given any maximal ideal M of $C(Z)$ the field $C(Z)/M$ contains a canonical copy of the real field $\mathbb{R}$ — the images of the constant functions under the canonical quotient map $C(Z) \xrightarrow{\nu} C(Z)/M$; a maximal ideal M is said to be *real*, if and only if the canonical map $\mathbb{R} \longrightarrow C(Z)/M$ is an isomorphism of ordered fields, and the following appears in [1, Theorem 5.14]:

Theorem 4.2. *The following are equivalent for any maximal ideal M of* $C(Z)$:

(a) *M is a real maximal ideal.*
(b) $\{Z(f) : f \in M\}$ *is an ultrafilter on* $\{Z(f) : f \in C(Z)\}$ *which is closed under countable intersections.*[5]
(c) $\{Z(f) : f \in M\}$ *has the countable intersection property.*

Finally it is known that there exists a one-to-one correspondence between the maximal ideals of $C(Z)$ and the ultrafilters on $\{Z(f) ; f \in C(Z)\}$, see [1, Chapter 2].

(2) An alternate description of realcompact spaces appear in [1, §5.9]:

Theorem 4.3. *A topological space X is realcompact, if and only if the real maximal ideals of* $C(Z)$ *are precisely the fixed maximal ideals of* $C(Z)$.

Thus it seems natural to investigate into the relationship between sober Borel spaces and realcompact topological spaces with the canonical Borel structure generated by the open subsets.

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[3] A topological space is said to be *realcompact*, if and only if it is homeomorphic to a closed subset of a product of the real line $\mathbb{R}$.

[4] A *frame* is a complete lattice L in which the finite meets distribute over arbitrary joins.

[5] For any $f \in C(Z)$, $Z(f) = \{z \in Z : f(z) = 0\}$, the set of all zeros of f.

Every frame seen as σ-frame admits a Boolean reflection, see [8]. So if L be a frame, and $\mathcal{B}L$ be the Boolean reflection of L, viewed as a σ-frame, then one has two notions of spatiality: one is the usual spatiality of the frame L as described in [5] or [6], and another is the spatiality of the Boolean σ-frame $\mathcal{B}L$ as described in Theorem 2.3. It seems quite natural to investigate the relationship between these two kinds of spatiality for a frame.

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